APPENDIX D (formerly E)

SAMPLE MATHEMATICS PROBLEMS

The sample problems in this appendix supplement those appearing in Chapter 2. The problems for kindergarten through grade seven are organized by grade level and according to the standards for the five strands in the *Mathematics Content Standards.* The problems for grades eight through twelve are presented according to the standards for the discipline headings algebra, geometry, and so forth. Teachers may use these problems as a resource to develop students' skills in working with the standards. [Complete citations for the sources following some of the mathematics problems in this appendix appear in "Works Cited." Many of the problems come from or are adapted from materials that are a part of the Third International Study of Mathematics and Science (TIMSS). TIMSS offers both a resource kit, *Attaining Excellence: A TIMSS Resource Kit,* and a Web site <http://www.csteep.bc.edu/TIMSS1/pubs_main.html>. *Ed.*]

Kindergarten

Number Sense

1.1, 1.2 How many students are in your class? How many chairs are in the room? Are there more chairs than students? What happens when there are more students than chairs?

1.2 Make sure that students can count forward accurately before presenting a series like the following:

30, 29, 28, __, __, 25, __, __, __, __, 20, 19, __, __, __, 15.

Grade One

Number Sense

1.2 Prove or disprove a classmate's claim that, "29 is more than 41 because 9 is more than 4 or 1." (This problem also applies to Mathematical Reasoning Standard 2.1.)

Grade Two

Number Sense

2.2 Find a three-digit number such that the sum of its digits is equal to 26. How many such numbers can you find?

3.2 Pretend you are at a store and you have $2.00 to spend. A pen costs 79 cents, a notepad 89 cents, and an eraser 49 cents. Suppose you want to buy two items and have the most money left over, which two would you buy? What is the largest number of pens you can buy? Notepads? Erasers? Explain how you know.

5.2 How many pennies does it take to make $1.57? How many nickels does it take to make $2.65?

Measurement and Geometry

1.1 Which is longer: the width of your classroom or 8 times the length of your desk?

Statistics, Data Analysis, and Probability

1.1 A kite has four panels. You have been asked to color it with either red or blue on each panel. How many different color kites can you make?

2.1 Look at these numbers: 50, 46, 42, 38, 34, 30, . . . There are many patterns that can produce these numbers. Please describe one. (Teacher: follow up with a question about which method is the simplest.) (Adapted from TIMSS gr. 4, U-4)

Grade Three

Number Sense

2.4 There are 54 marbles. They are put into 6 bags so that the same number of marbles is in each. How many marbles would 2 bags contain? (Adapted from TIMSS gr. 4, K-9)

2.8 Here is a number sentence: 2,000 + ___ + 30 + 9 = 2,739. What number should go where the blank is to make the sentence true? (TIMSS gr. 4, S-2)

3.1 Janis, Maija, and their mother were eating a cake. Janis ate 1/2 of the cake. Maija ate 1/4 of the cake. Their mother ate 1/4 of the cake. How much of the cake is left? (Adapted from TIMSS gr. 8, P-14)

3.1 Sam, who is 6 years old, likes vanilla ice cream with his apple pie. Sam said that 1/3 of an apple pie is less than 1/4 of the same pie. Is Sam correct in his estimate? (Adapted from TIMSS gr. 4, V-1)

Measurement and Geometry

1.2 Make an outline of your hand with your fingers together on a piece of grid paper. Assuming that each grid is 1 cm^2, what is roughly the area of your hand?

Grade Four

Algebra and Functions

1.4 Maria and her sister, Louisa, leave home at the same time and ride their bicycles to school 9 kilometers away. Maria rides at a rate of 3 kilometers in 10 minutes. How long will it take her to get to school? Louisa rides at a rate of 1 kilometer in 3 minutes. How long will it take her to get to school? Who arrives first? (Adapted from TIMSS gr. 4, U-3)

1.5 My plane was supposed to leave San Francisco at 8:42 a.m. and arrive in Los Angeles at 9:55 a.m. But it started 11 minutes late, and to make up for lost time, the pilot increased the speed and shortened the flight time to 58 minutes. What time did I arrive in Los Angeles? (This problem also applies to Mathematical Reasoning Standard 1.1.)

2.1 What is the remainder when 1,200,354,003 is divided by 5?

Measurement and Geometry

1.2, 1.4 Given 12 square tiles, all the same size, describe all the rectangles you can that use all the tiles. Find the perimeter of each rectangle.

3.7 Assume that the sum of the length of any two sides of a triangle is greater than the length of the third side. If the lengths of the sides of a triangle are required to be whole numbers, how many such triangles are there with a perimeter of 14? List all of them.

Statistics, Data Analysis, and Probability

1.1 If six people enter a room and shake hands with each other once, how many handshakes occur?

Grade Five

Number Sense

1.2 Change to decimals: $\frac{17}{1{,}000}$, $\frac{3}{20}$, 6%, 35½

1.2 Change to fractions: 0.03, 1.111, 8%, 21

1.2 Change to percents: 0.07, 0.165, $\frac{17}{20}$, $\frac{1}{8}$

1.2 6 is what % of 25?

1.2 What is 15% of 44?

1.2 30 is 20% of what?

1.2 Betty paid $23.60 for an item that was reduced by 20%.

1. What was the original price?
2. If the original price was reduced by 25%, what is the sale price?

1.4 Write as a product of primes using exponents (use factor trees or equivalents):

18, 48, 100

1.3, 1.4 What is the largest square of a whole number that divides 48? What is the largest cube of a whole number that divides 48?

1.5 Arrange in order from smallest to largest:

¾, 25%, 0.3, 2½, 0.295

10173 **2.1** Find the average of 6.81, 7, 5.2 and round the answer to the nearest
10174 hundredth.

10175 **2.1** Evaluate 0.25 (3 – 0.75).

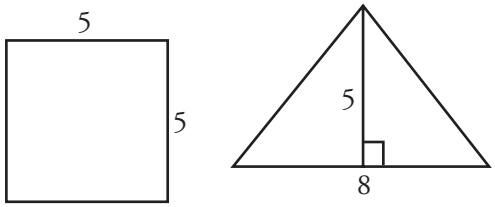
10176 **2.5** Do the following problems mentally.

10177 $\left(\frac{9,185}{2,117} \times \frac{12}{13}\right) \div \frac{9,185}{2,117} = ?$

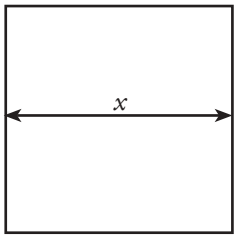
10178 $\left(\frac{9,185}{13} \times \frac{12}{2,117}\right) \div \frac{9,185}{2,117} = ?$

10179 **Measurement and Geometry**

10180 **1.0** Find the areas (dimensions are in cm):

10181 

10182 **1.1** How many segments *x* will fit on the perimeter of the square?

10183 

10184 **1.3, 1.4** Determine the volume of a rectangular solid with base 65 cm, height 70 cm,
10185 and width 50 cm. For the same rectangular solid, determine its surface
10186 area. (Make sure that your answer is expressed in the correct units.)

10187 **1.4** Identify the relevant dimension as length, area, or volume:

10188 1. The perimeter of a triangle

10189 2. The capacity of a barrel

3. The capacity of a box
4. The amount of sod needed to cover a football field
5. The number of bricks needed to pave a path
6. The height of a tree

2.1 Explain how to make the following basic constructions with a straight edge and compass; e.g., an equilateral triangle, a regular hexagon, a line passing through a given point and perpendicular to a given line.

2.2 Find the third angle of a triangle if you know that one angle is 60° and the second angle is 20°.

Statistics, Data Analysis, and Probability

1.2. Draw a circle graph to display the following data: A certain municipal district spends 6 million dollars per year—$2,507,000 on education, $1,493,000 for public safety, $471,000 for libraries, $536,000 for road maintenance, and $993,000 for miscellaneous expenses. (This problem also applies to Number Sense Standards 1.1 and 1.2.)

Grade Six

Number Sense

1.2, 1.3 Complete the following statements:

1. If 3 ft. = 1 yd., then 7 ft. = ? yd.
2. If 32 oz. = 1 qt., then 6.7 qt. = ? oz.

1.2, 1.3 In a lemonade punch, the ratio of lemonade to soda pop is 2:3. If there are 24 gallons of punch, how much lemonade is needed?

2.1 Find the sum $\frac{5}{6}+\frac{3}{10}$.

2.3, 2.4 Write the following as an integer over a whole number:

8, –6, $4\frac{1}{2}$, $-1\frac{1}{5}$, 0, 0.013, –1.5

2.4

1. Find the least common multiple of 6 and 10 (count by sixes until you come to a multiple of 10).
2. List the first 20 multiples of 6.
3. List the first 20 multiples of 10.
4. List all the multiples that six and 10 have in common that are less than or equal to 120.

2.4

1. Make a sieve of Eratosthenes up to 100.
2. Find the greatest common factor of 18 and 30 (list all factors of 18 until you come to a factor of 30).
3. Reduce $\frac{18}{30}$.

Algebra and Functions

1.2 Moe was paid $7 per hour and earned $80.50. How many hours did Moe work?

1.2 Write the following in symbolic notation using *n* to represent the number:

1. A number increased by 33
2. The product of a number and (–7)
3. $8\frac{1}{2}$ decreased by some number
4. The square of some number which is then divided by 7

5. The sum of some number and $\frac{1}{3}$ which is then increased by the third power of the same number

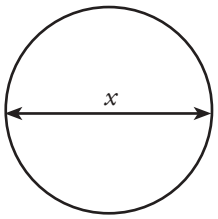
1.2, 3.1 A rectangle is constructed with 8 feet of string. Suppose that one side is $1\frac{4}{14}$ feet long. What is the length of the other side?

1.3 True or false?

$(25 + 16) \times 6 = 25 + 16 + 6$.

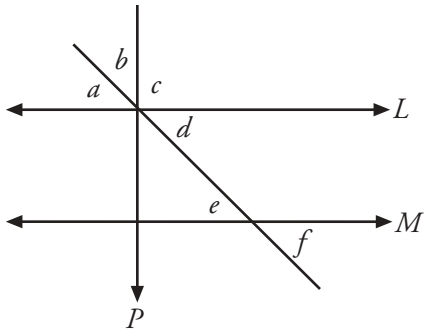
Measurement and Geometry

1.2 How many segments *x* will fit on the circumference of the circle?



1.1, 1.3 Use the formula $\pi r^2 h$ for the volume of a right circular cylinder. What is the ratio of the volume of such a cylinder to the volume of one having half the height but the same radius? What is the ratio of the volume of such a cylinder to the volume of one having the same height but half the radius? (This problem also applies to Number Sense Standard 1.2.)

2.1 Line *L* is parallel to line *M*. Line *P* is perpendicular to *L* and *M*. Name the following. If none can be named, leave the space blank.



1. Complementary _____

2. Supplementary _____

3. Vertical _____

4. Alternate interior _____

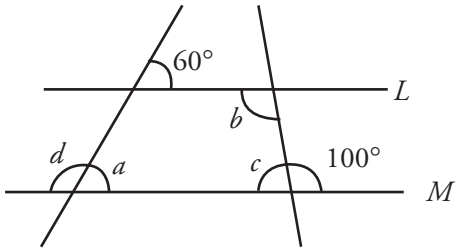
5. Corresponding _____

6. Acute _____

7. Right _____

8. Obtuse _____

2.2 Line *L* is parallel to line *M*. Find the missing angles.



Statistics, Data Analysis, and Probability

2.2, 2.5 Fifty red marbles are placed in a box containing an unknown number of green marbles. The box is thoroughly mixed, and 50 marbles are taken out. Ten of those marbles are red. Does this imply that the number of green marbles was 200?

3.1, 3.4 Make a tree diagram of all the possible outcomes of four successive coin tosses. How many paths in the tree represent two heads and two tails? Suppose the coin is weighted so that there is a 60% probability of heads with each coin toss. What is the probability of one head and one tail?

Grade Seven

Number Sense

1.1 Convert to scientific notation; compute and express your answer in scientific notation and in decimal notation.

1. $\dfrac{(350,000)(0.0049)}{.25}$

2. $\dfrac{(0.000042)(0.0063)}{((140,000)(70,000)(0.18))}$

1.6 Peter was interested in buying a basketball. By the time he saved enough money, everything in the sporting goods store had been marked up by 15%. Two weeks later, however, the same store had a sale, and everything was sold at a 15% discount. Peter immediately bought the ball, figuring that he was paying even less than before the prices were raised. Was he mistaken?

1.7 What will be the monthly payments on a loan of $50,000 at 12% annual interest so that it will be paid off at the end of 10 years? How much total interest will have been paid? Do the same problem with 8% annual interest over 10 years. Do the same problem with 10% annual interest over 15 years. Solve the problem using simple interest. (Use calculators.)

2.2 Reduce $\dfrac{910}{1,859}$.

2.2 Subtract and reduce to lowest terms:

$$\left(\frac{81}{143}\right)-\left(\frac{7}{208}\right)$$

(For clarification see the discussion in Appendix A on the addition and subtraction of fractions.)

2.4 Determine without a calculator which is bigger: $\sqrt{291}$ or 17?

2.5 Consider two numbers *A* and *B* on the number line. Determine which is larger: the distance between *A* and *B* or the distance between |A| and |*B*|? Always? Sometimes? Never?

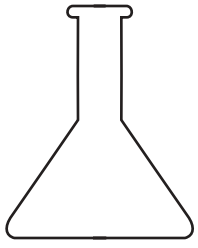
Algebra and Functions

1.1 Gabriel bought a CD player, listed at $*a*, at a 20% discount; he also had to pay an 8% sales tax. After three months he decided that its sound quality was not good enough for his taste, and he sold it in the secondhand market for $*b*, which was 65% of what he paid originally. Express *b* as a function of *a*.

1.1, 4.2 A car goes 45 mph and travels 200 miles. How many hours will it take for the car to reach its destination?

1.1, 4.2 A plane flying at 450 mph leaves San Francisco. One-half hour later a second plane flying at 600 mph leaves, flying in the same direction. How long will it take the second plane to catch the first? How far from San Francisco will this event happen?

1.5 Water is poured at a constant rate into a flask shaped like the one in the illustration that follows. Draw a graph of the water level in the flask as a function of time.



2.1, 2.2 Simplify to a monomial, or reduce to a single monomial.

1. $\frac{x^5}{x^3}$

2. $\frac{x^3}{x^5}$

3. $\frac{x^5}{x^5}$

4. $\frac{\left(42a^5b^3\right)}{\left(14a^2b^9\right)}$

5. $\frac{x^{-8}}{x^{-7}}$

6. $\frac{\left(a^7b^{-3}c^9\right)}{\left(a^4b^{-3}c^{10}\right)}$

3.1, 3.2 Write the equation of the surface area of a cube of side length *x*. Graph the surface area as a function of *x*.

3.1 The amount of paint needed to paint over a surface is directly proportional to the area of the surface. If 2 quarts of paint are needed to paint a square with a side of 3 ft., how many quarts must be purchased to paint a square whose side is 4 ft. 6 in. long?

3.4 What is the slope of the straight line which is the graph of the function expressing the length of a semicircle as a function of the radius?

4.1 Becky and her sister have some money. The ratio of their money is 3:1. When Becky gives $5 to her sister, their ratio will be 2:1. How much money does Becky have? (World Math Challenge 1995)

4.2 Three people set out on a car race to see who would be the first to get to town T and back. Anne maintained a steady speed of 80 mph throughout the race. Lee averaged 90 mph on the way out, but he could manage only an average of 70 mph on the way back. Javier started slowly and averaged

65 mph during the first third of the race, but he increased his speed to 85 mph in the second third and finished with a blazing 100 mph in the last third. Who won?

(*Note:* This is a difficult problem that would be particularly good for advanced students.)

Measurement and Geometry

1.1 Know the following approximations:

1. 1 meter ≈ 1 yard (baseball bat)
2. 1 cm ≈ ½ inch (width of a fingernail)
3. 1 km ≈ .6 miles
4. 1 kg ≈ 2.2 lbs. (a textbook)
5. 1 liter ≈ 1 quart
6. 1 gram ≈ (1 paper clip)
7. 1 mm ≈ (thickness of a dime)

1.3 A bucket is put under two faucets. If one faucet is turned on alone, the bucket will be filled in 3 minutes; if the other is turned on, the bucket will be filled in 2 minutes. If both are turned on, how many seconds will it take to fill the bucket?

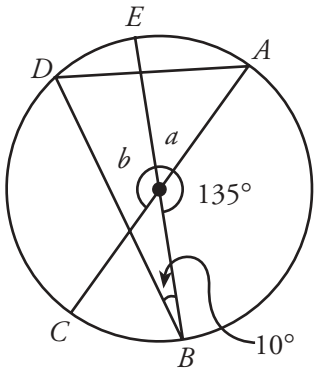
2.1 Compute the area and perimeter of a regular hexagon inscribed in a circle of radius 2.

2.1 Compute the volume and surface area of a square-based pyramid whose lateral faces are equilateral triangles with each side equal to 4.

3.2 Determine the vertices of a triangle, whose vertices were originally at (1, 2), (–3, 0), and (–1, 5), after it is translated 2 units to the right and 1 unit down and then reflected across the graph of $y = x - 3$.

3.3 What is the distance from the center of a circle of radius 3 to a chord of length 5 cm?

3.1 Find the missing angles and arcs. ($\angle B$ is 10°.)



3.2

1. Minor $\overset{\frown}{AB}$
2. Angle *a*
3. Major $\overset{\frown}{\ }$
4. Angle *b*
5. $\overset{\frown}{EC}$
6. $\overset{\frown}{CB}$
7. $\overset{\frown}{ED}$
8. $\overset{\frown}{DC}$

Grades Eight Through Twelve

Algebra I

1.0, 24.0 Prove or give a counterexample: The average of two rational numbers is a rational number.

2.0 I start with a number and apply a four-step process: I (1) add 13; (2) multiply by 2; (3) take the square root; and (4) take the reciprocal. The result is 1/4. What number did I start with? If I start with the number *x*, write a formula that gives the result of the four-step process.

2.0 What must be true about a real number *x* if $x = \sqrt{x^2}$?

2.0 Write as a power of *x:* $\frac{\sqrt{x}}{x \cdot \sqrt[3]{x}}$.

2.0 Solve for *x:* $x^3 = \frac{1}{2\sqrt{2}}$.

3.0 Solve for *x:* $3x + 2 = 14$.

3.0 Express the solution using interval notation: $x + 1 \geq 2.$

Sketch the interval in the real number line that is the solution for $x - 5 < 2.$

4.0 Expand out and simplify $2(3x + 1) - 8x$.

4.0 Solve for *x* in each case*:*

$$5x - 2 \leq -3(x + 1) + 2$$

$$2 - \left(2 - (3x + 1) + 1\right) = 3(x - 2) + x$$

$$\frac{3}{x - 2} = \frac{4}{x + 5}$$

5.0 To compute the deduction that you can take on your federal tax return for medical expenses, you must deduct 7.5% of your adjusted gross income from your actual medical expenses. If your actual medical expenses are

\$1,600 and your deduction is less than \$100, what can you conclude about your adjusted gross income?

5.0 Joe is asked to pick a number less than 100, and Moe is asked to guess it. Joe picks 63. Write an inequality that says that Moe's guess is within 15, inclusive, of the number Joe has in mind. Solve this inequality to find the range of possibilities for Moe's guess.

5.0 Four more than three-fifths of a number is 24. Find the number.

5.0 Luis was thinking of a number. If he multiplied the number by 7, subtracted 11, added 5 times the original number, added -3, and then subtracted twice the original number, the result was 36. Use this information to write an equation that the number satisfies and then solve the equation.

6.0 The cost of a party at a local club is \$875 for 20 people and \$1,100 for 30 people. Assume that the cost is a linear function of the number of people. Write an equation for this function. Sketch its graph. How much would a party for 26 people cost? Explain and interpret the slope term in your equation.

6.0 Graph $2x - 3y = 4$. Where does the line intersect the *x*-axis? Where does the line intersect the *y*-axis? What is the slope?

6.0 Sketch the region in the *x*-*y* plane that satisfies both of the following inequalities: $y < 3x + 1,\ 2x + 3y + 8 > 0$

6.0, 7.0 Find an equation for the line that passes through (2, 5) and (-3, 1). Where does the line intersect the *x*-axis? Where does the line intersect the *y*-axis? What is the slope?

6.0, 7.0 Find an equation for the line that passes through (5, 3) and (5,−2). Where does the line intersect the *x*-axis? Where does the line intersect the *y*-axis? What is the slope?

7.0 The weight of a pitcher of water is a linear function of the depth of the water in the pitcher. When there are 2 inches of water in the pitcher, it weighs 2 lbs.; and when there are 8 inches of water in the pitcher, it weighs 5 lbs. Find a formula for the weight of the pitcher as a function of the depth of the water.

7.0 Find an equation for the line that passes through (−2, 5) and has slope −2/3.

8.0 Find the equation of the line that is perpendicular to the line through (2, 7) and (−1, 3) and passes through the *x*-intercept of that line.

8.0 Are the following two lines perpendicular, parallel, or neither?

$2x + 3y = 5$

$3x + 2y - 1 = 0$

8.0 If the line through (1, 3) and (*a,* 9) is parallel to $3x - 5y = 2$, what is *a*?

9.0 Line 1 has equation $3x + 2y = 3$, and line 2 has equation $-2x + y = 5$. Find the point of intersection of the two lines.

9.0 Sketch a graph of the values of *x* and *y* that satisfy both of the inequalities:

$3x + 2y \geq 3$

$-2x + y \leq 5$

10.0 The volume of a rectangular prism with a triangular base is $36m^3 - 72m^2 + 29m - 3$. Assume that the height of the prism is $3m - 1$ and the height of the triangle is $6m - 1$. What is the base of the triangle?

10437 **10.0** Simplify $[(3b^2 - 2b + 4) - (b^2 + 5b - 2)]\ (b + 2)$.

10438 **11.0** Solve for *x:* $\frac{x^2 - 4}{x - 2} + x^2 - 4 = 0.$

10439 **12.0** Reduce to lowest terms: $\frac{x^3 + x^2 - 6x}{x^2 + 13x + 30}$.

10440 **12.0** Solve for *x*: $\frac{3}{x - 1} + \frac{10}{x^2 - 2x + 1} = 4.$

10441 **13.0** Solve for *x:* $\frac{\frac{x + 2}{x - 3} \cdot \frac{x^2 + 5x - 24}{x - 6}}{x + 8} + 3 = 0.$

10442 **14.0** Where does the graph of $f(x) = \frac{x^3 + 2x^2 - 15x}{x + 1}$ intersect the *x*-axis?

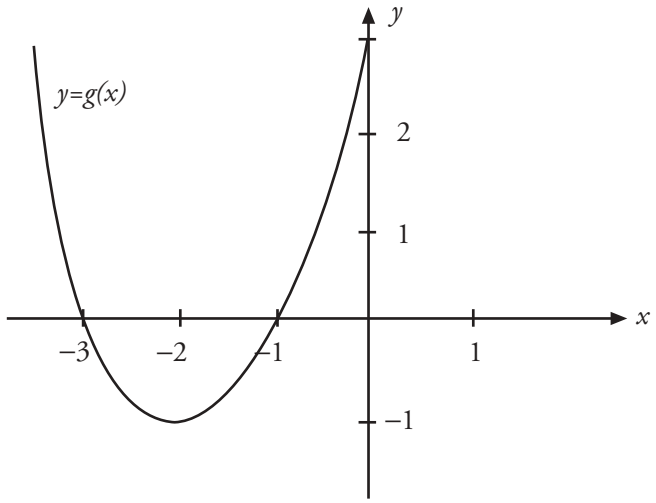
10443 **15.0** Mary drove to work on Thursday at 40 mph and arrived one minute late.
10444 She left at the same time on Friday, drove at 45 mph, and arrived one
10445 minute early. How far does Mary drive to work?

10446 **15.0** Suppose that peanuts cost $.40/lb. and cashews cost $.72/lb. How many
10447 pounds of each should be used to make an 80 lb. mixture that costs
10448 $.48/lb.?

10449 **16.0** The following points lie on the graph of a relation between *x* and *y*:
10450 (0, 0), (–2, 3), (3, –2), (2, 3), (–3, –3), (2, –2)
10451 Can *y* be a function of *x*? Explain. Can *x* be a function of *y*? Explain.

10452 **17.0** Determine the domain of the function $f(x) = \sqrt{x - 6}.$

10453 **17.0** Determine the range of the function *g* whose graph is shown below.



10454

10455 Let $f(x) = x^2 - 16$ (in words *x* squared minus 16), and *x* is a real number.

10456 **4.0** What is the domain of $f(x)$?

10457 **5.0** What is the range of $f(x)$?

10458 **6.0** For what values of *x* is $f(x)$ negative?

10459 **7.0** What are the domain and range of square root $(x^2 - 16)$ when *x* is assumed

10460 to be a real number?

10461 **18.0** Does the equation $x^2 + y^2 = 1$ determine *y* as a function of *x*? Explain.

10462 **20.0** Solve for *x:* $2x^2 - 3x - 5 = 0$.

10463 **20.0** Let $f(x) = ax^2 + bx + c$. Suppose that $b^2 - 4ac > 0$. Use the quadratic formula

10464 to show that *f* has two roots.

10465 **22.0** At how many points does the graph of $g(x) = 2x^2 - x + 1$ intersect the *x*-axis?

23.0 A ball is launched straight up into the air from the ground at a rate of 64 feet per second. Its height h above the ground (in feet) after t seconds is $h = 64t - 16t^2$.

How high is the ball after 1 second? When is the ball 64 feet high? For what values of t is $h = 0$? What events do these represent in the flight of the ball?

23.0 The braking distance of a car (how far it travels after the brakes are applied until it comes to a stop) is proportional to the square of its speed. Write a formula expressing this relationship and explain the meaning of each term in the formula. If a car traveling at 50 mph has a breaking distance of 105 feet, then what would its braking distance be if it were traveling 60 mph? (ICAS 1997)

24.0 Provide numbers to show how the following can be false and if possible describe when it is true:

$\sqrt{a^2 + b^2} < a + b$ whenever $a \geq 0$ and $b \geq 0$

25.0 Suppose that 9 is a factor of xy, where x and y are counting numbers. At least one of the following is true. Which of the following statements are necessarily true? Explain why.

a. 9 must be a factor of x or of y.

b. 3 must be a factor of x or of y.

c. 3 must be a factor of x and of y.

25.0 A problem is given, to find all solutions to the equation $(2x + 4)^2 = (x + 1)^2$. Comment on any errors in the following proposed solutions:

$$(2x + 4)^2 = (x + 1)^2$$

Take the square root of both sides to find $2x + 4 = x + 1$

Subtract *x* and 4 from both sides to obtain $2x + 4 - x - 4 = x + 1 - x - 4$

Simplify to conclude $x = -3$

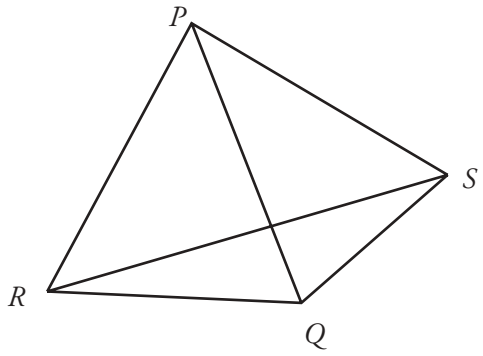
Geometry

3.0 Prove or disprove: Any two right triangles with the same hypotenuse have the same area.

3.0 True or false? A quadrilateral is a rectangle only if it is a square.

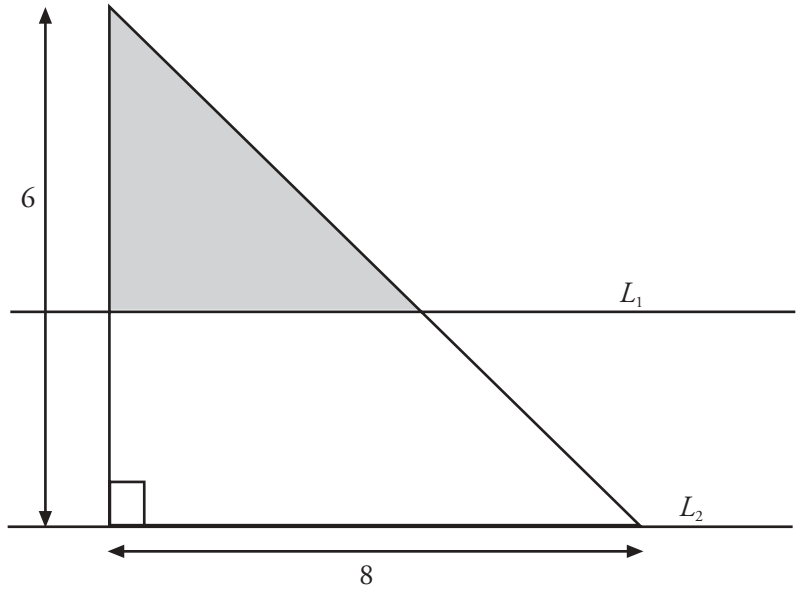
3.0 Suppose that all triangles that satisfy property *A* are right triangles. Is the following statement true or false? A triangle that does not satisfy the Pythagorean theorem does not satisfy property *A.*

4.0 Suppose that triangle *PRS* is isosceles, with $\overline{RP} = \overline{PS}$. Show that if the segment *PQ* bisects the $\angle RPS$, then $RQ = QS$.



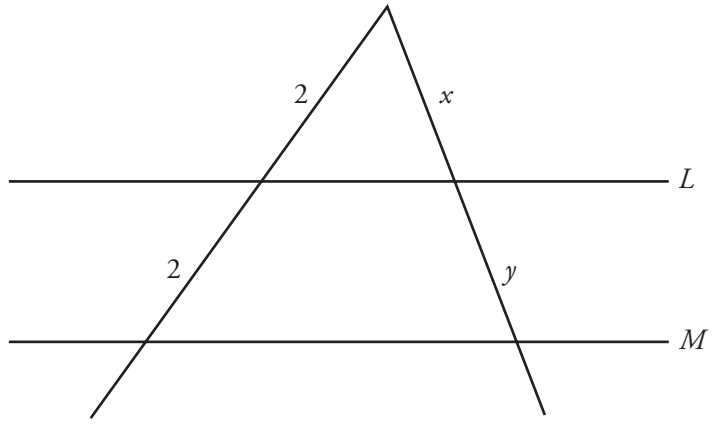
4.0 Suppose that *R* and *S* are points on a circle. Prove that the perpendicular bisector of the line segment *RS* passes through the center of the circle.

5.0 In the figure shown below, the area of the shaded right triangle is 6. Find the distance between the parallel lines, L_1 and L_2. Explain your reasoning.



6.0 Using a geometric diagram, show that for any positive numbers *a* and *b,* $\sqrt{a^2+b^2} < a+b$.

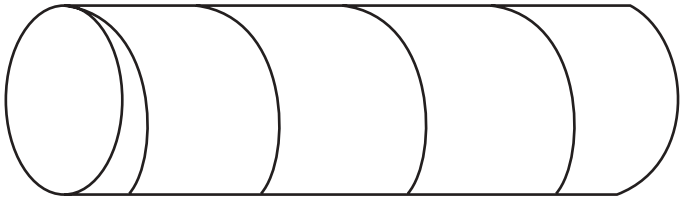
7.0, 4.0 On the following diagram, with distances as shown, prove that if $x = y$, then the lines *L* and *M* are parallel:



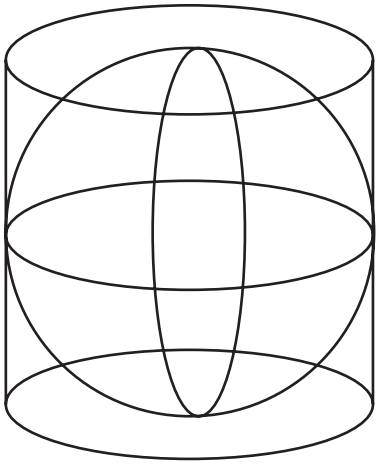
7.0 Prove that if a diagonal of a parallelogram bisects an angle of a parallelogram, then the parallelogram is a rhombus.

7.0 Prove that if the base angles of a trapezoid are congruent, then the trapezoid is isosceles.

8.0 A string is wound, evenly-spaced, around a circular rod. The string goes exactly one time around the rod. The circumference of the rod is 4 cm, and its length is 12 cm. Find the length of the string. What is the length of the string if it goes exactly four times around the rod? (Adapted from TIMSS)



9.0 A sphere of radius 1 can be inscribed in a cylinder so that it touches the top face, bottom face, and intersects the lateral face in a circle. Find the volume of the cylinder.



9.0 A right prism with a 4-inch height has a regular hexagonal base. The prism has a volume of 144 cubic inches. Find the surface area of the prism.

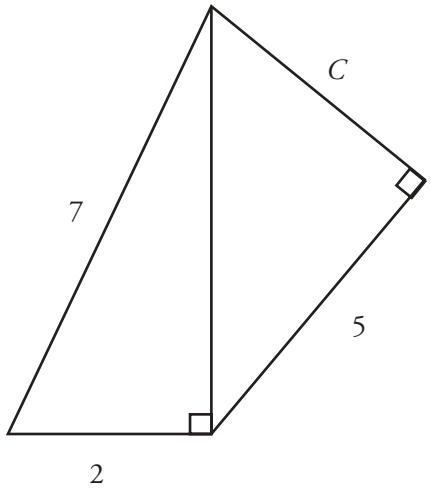
10.0 A trapezoid with bases of length 12 and 16 is inscribed in a circle of radius 10. The center of the circle lies inside the trapezoid. Find the area of the trapezoid.

11.0 Brighto soap powder is packed in cube-shaped cartons. A carton measures 10 cm on each side. The company decides to increase the length of each edge of the carton by 10 percent. How much does the volume increase? (TIMSS)

12.0 A regular polygon has exterior angles, each measuring 10 degrees. How many sides does the polygon have?

13.0 Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

15.0 Find the length of the side labeled *C* in the figure shown below:



15.0 The bottom of a rectangular box is a rectangle with a diagonal whose length is $4\sqrt{3}$ inches. The height of the box is 4 inches. Find the length of a diagonal of the box.

16.0 Given a circle, use an unmarked straightedge and a compass to find the center of the circle.

17.0 The vertices of a triangle *PQR* are the points *P*(1, 2), *Q*(4, 6), and *R*(–4, 12). Which one of the following statements about triangle *PQR* is true?

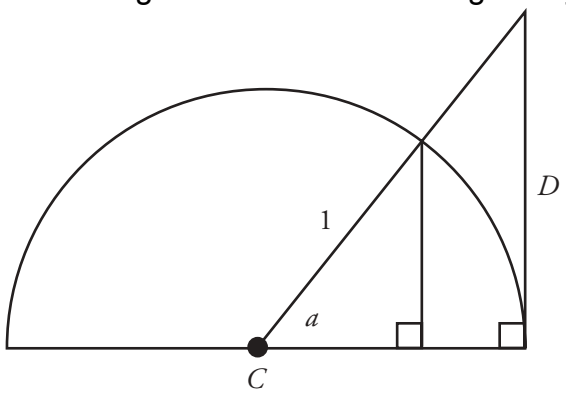
1. *PQR* is a right triangle with right $\angle P$.

2. *PQR* is a right triangle with right $\angle Q$.

3. *PQR* is a right triangle with right $\angle R$.

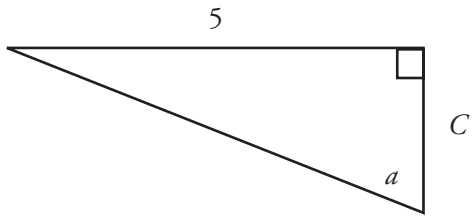
4. *PQR* is not a right triangle. (TIMSS)

18.0 Shown below is a semicircle of radius 1 and center *C.* Express the unknown length *D* in terms of the angle *a* by using a trigonometric function.



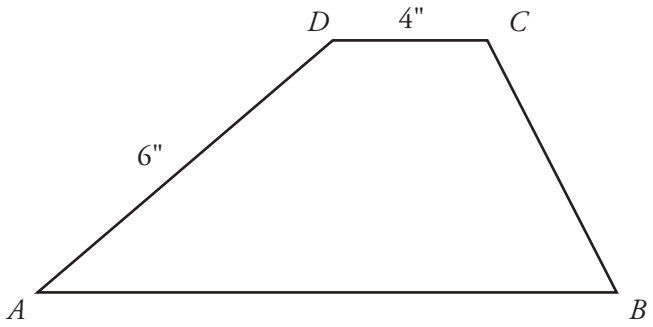
18.0 If α is an acute angle and $\cos \alpha = \frac{1}{3}$, find $\tan \alpha$

19.0 Find the length of side *C* below, if $\angle a$ measures 70 degrees:

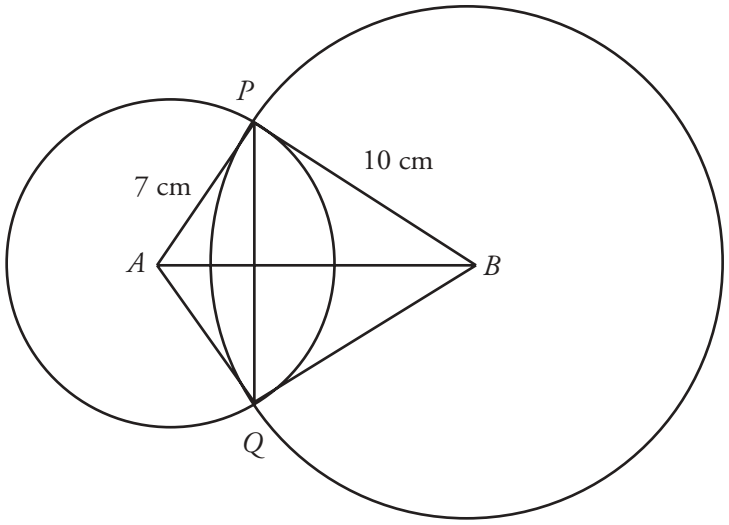


20.0 Each side of the regular hexagon *ABCDEF* is 10 cm long. What is the length of the diagonal *AC?* (TIMSS)

20.0 Express the perimeter of the trapezoid *ABCD* in the simplest exact form. Angle *DAB* measures 30 degrees, and angle *ABC* measures 60 degrees.



21.0 Two circles with centers *A* and *B,* as shown below, have radii of 7 cm and 10 cm, respectively. If the length of the common chord *PQ* is 8 cm, what is the length of *AB?* Show all your work. (TIMSS)

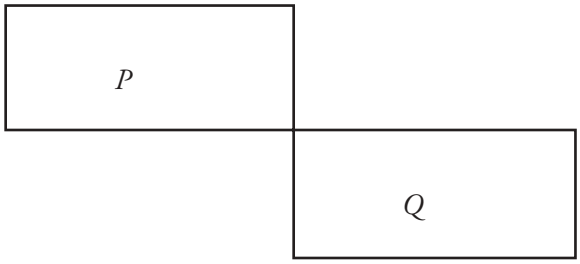


22.0 A translation maps $A = (2, 3)$ onto $A' = (3, 5)$. Under the same translation, find the coordinates of *B′,* the image of *B* (1, 4). (TIMSS)

22.0 Which response listed below applies to the statement that follows? The rectangle labeled *Q* cannot be obtained from the rectangle *P* by means of:

1. Reflection (about an axis in the plane of the page)
2. Rotation (in the plane of the page)
3. Translation
4. Translation followed by a reflection (TIMSS)

DRAFT: January 28, 2005



10574

10575 **Algebra II**

10576 **1.0** Express the solution using interval notation:

10577 $|2x - 3| > 4$

10578 **1.0** Sketch the interval in the real number line that is the solution for:

10579 $\frac{x-3}{2} > 5$

10580 **2.0** Solve the system of linear equations:

10581 $x + 2y = 0$

10582 $x + z = -1$

10583 $y - z = 2$

10584 **4.0** Simplify $\frac{x^3 - y^3}{x^2 - y^2}$.

10585 **4.0** Simplify $\frac{\sqrt{x} + y}{x - y^2}$.

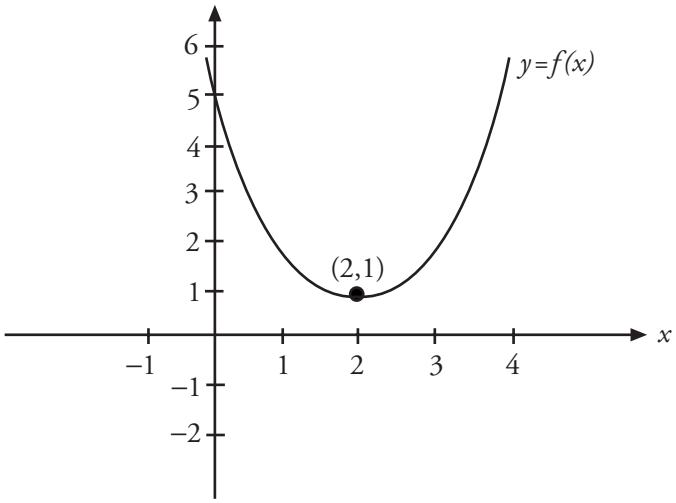
10586 **5.0** Locate all complex solutions to $z^2 + 4$ in the complex plane.

10587 **6.0** Write in the form $a + bi$, where i is a square root of -1:

10588 $\frac{(3-2i)^2}{2+i}$

10589 **8.0** Find all solutions to the equation $x^2 + 5x + 8 = 0$.

9.0 The function $f(x) = (x - b)^2 + c$ is graphed below. Use this information to identify the constants b and c. . The minimum value of the function occurs when $x = 2$, and $f(2) = 1$.



10.0 Graph the function $f(x) = 2(x + 3)^2 - 4$ and determine the minimum value for the function.

10.0 Find the vertex for the graph of $f(x) = 3x^2 - 12x + 4$.

11.0 Solve for x in each of the following and explain each step:

$$\log_3(x+1) - \log_3 x = 1$$

$$\log_{\sqrt{b}} 7 = \log_b x$$

12.0 Scientists have observed that living matter contains, in addition to common carbon, C12, a fixed percentage of a radioactive isotope of carbon, C14. When the living material dies, the amount of C12 present remains constant, but the amount of C14 decreases exponentially with a half life of 5,550 years. In 1965 the charcoal from cooking pits found at a site in Newfoundland used by Vikings was analyzed, and the amount of C14 remaining had decreased to 88.6 of the amount present when the charcoal

was fresh. What was the approximate date of this Viking settlement? (Adapted from ICAS 1997)

13.0 Simplify to find exact numerical values for:

$\log_{\sqrt{b}}(b^2)$

$b^{3\log_b 2 - \log_b 5}$

14.0 Write as a single logarithm $\frac{\log_3 7}{\log_3 5}$.

15.0 Is the following true for all real numbers *x,* for some real numbers *x,* or for no real numbers *x*?

$$\frac{\sqrt{(1-x^2)^2}}{1-x} = 1+x$$

16.0 If $xy = 1$ and x is greater than 0, which of the following statements is true?

1. When x is greater than 1, y is negative.
2. When x is greater than 1, y is greater than 1.
3. When x is less than 1, y is less than 1.
4. As x increases, y increases.
5. As x increases, y decreases. (TIMSS)

17.0 Write in standard form the conic section whose equation is given by $4x^2 - 8x - y^2 + 4y = 4$ to determine whether it is a parabola, a hyperbola, or an ellipse.

18.0 An examination consists of 13 questions. A student must answer only one of the first two questions and only nine of the remaining ones. How many choices of questions does the student have? (Adapted from TIMSS)

19.0 A lottery will be held to determine which three members of a club will attend the state convention. This club has 12 members, 5 of whom are women. What is the probability that none of the representatives of the club will be women?

20.0 Determine the middle term in the binomial expansion of $\left(x-\frac{2}{x}\right)^{10}$. (ICAS 1997)

21.0 Use mathematical induction to show that

$$1+2+3+4+\cdots+n=\frac{n(n+1)}{2}$$

22.0 Find the sum of the following infinite series:

$$\frac{3}{5}+\frac{9}{25}+\frac{27}{125}+\frac{81}{625}+\cdots$$

24.0 Sketch a graph of a function g that satisfies the following conditions: g does not have an inverse function, g(x) < x for all x, and g (2) > 0.

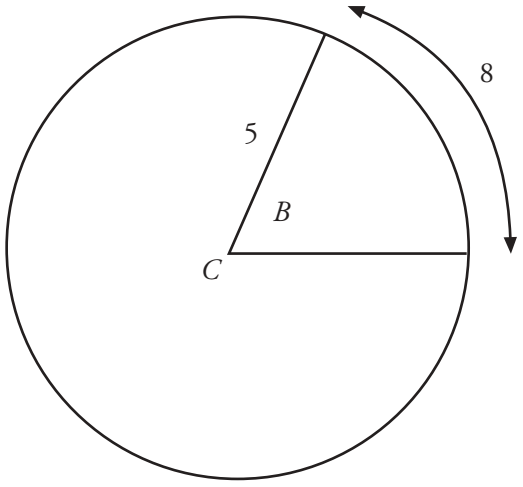
Trigonometry

1.0 Express in degrees:

π/5 radians

1/8 revolution

1.0 Find the indicated angle B in radians, if C is the center of the circle:



2.0 Graph the functions f(x) = sin x and g(x) = cos x, where x is measured

in radians, for x between 0 and 2π. Identify the points of intersection of the two graphs.

3.0 Prove that $\sec^2 x + \csc^2 x = \sec^2 x \cdot \csc^2 x$.

5.0 Use the definition of $f(x) = \tan(x)$ to determine the domain of *f.*

6.0 Identify all vertical asymptotes to the graph of $g(x) = \sec x$.

7.0 A line with positive slope makes an angle of 1 radian with the positive *x*-axis at the point (3, 0). Find an exact equation for this line.

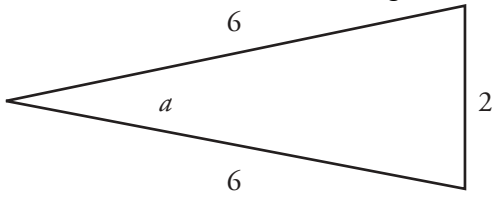
8.0 If $\tan(x) = \tan(\pi/5)$ and $3\pi < x < 4\pi$, find *x*.

8.0 Graph $f(x) = \sin x$ and the principal value of $g(x) = \sin^{-1} x$ on the same axes. Write a description of the relationship between the two graphs.

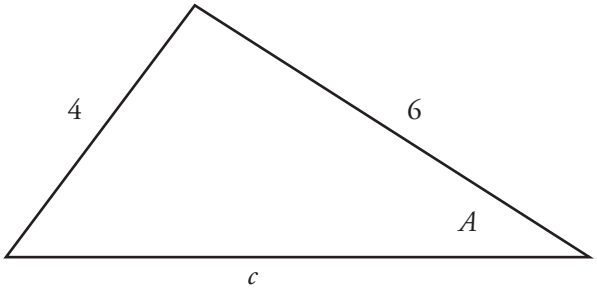
9.0 Find an angle α between 0 and $-\pi$ for which $\cos(\alpha) = -1/2$

11.0 Solve for θ, where $0 < \theta < 2\pi$: $(\cos\theta)(\sin 2\theta) - 2\sin\theta + 2 = 0$.

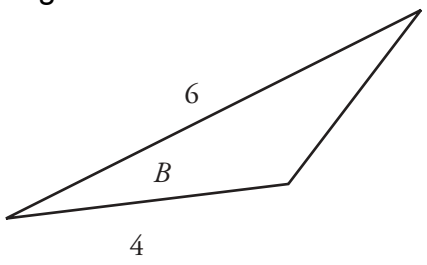
12.0 Find the measure of the angle *a* in the triangle below:



13.0 Solve for the distance *c* on the triangle shown below, if the angle *A* is 30°:



14.0 Find the area of the triangle shown below if the angle *B* measures 20 degrees:



15.0 Find all representations in polar coordinates of the point whose rectangular coordinates are $(2\sqrt{3}, -2)$.

17.0 Represent $i + 1$ in polar form. Use this to compute $(i + 1)^2$.

18.0 Find all square roots of *i.*

19.0 A person holds one end of a rope that runs through a pulley and has a weight attached to the other end. The section of rope between the person and the pulley is 20 feet long; the section of rope between the pulley and the weight is 10 feet long. The rope bends through an angle of 35 degrees in the pulley. How far is the person from the weight?

19.0 How long does it take for a minute hand on a clock to pass through 1.5 radians?

Mathematical Analysis

1.0 Find any points of intersection (first in polar coordinates and then in rectangular coordinates) of the graphs of $r = 1 + \sin\theta$ and the circle of radius 3/2 centered about the origin. Verify your solutions by graphing the curves. Find any points of intersection (first in polar coordinates and then in rectangular coordinates) of the graphs of $r = 1 + \sin\theta$ and the line with slope 1

that passes through the origin. Verify your solutions by graphing the curves. (ICAS 1997)

2.0 Compute $\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{11}$.

5.0 Consider the locus of points in the plane whose distance to (0, 1) is twice its distance from (0, −2). Identify this conic section and find its equation in standard form.

6.0 Sketch the graph of $f(x)=\frac{x}{x^2-4}$, showing all asymptotes.

7.0 Sketch a graph of the curve determined by the equations:

$x = \cos(t^2) + 1$

$y = \sin(t^2)$ for $0 < t < 5$

and find another set of parametric equations that describe the same curve.

Probability and Statistics

1.0 A warning system installation consists of two independent alarms having probabilities of 0.95 and 0.90, respectively, of operating in an emergency. Find the probability that at least one alarm operates in an emergency. (Adapted from TIMSS)

1.0 Arlene and her friend want to buy tickets to an upcoming concert, but tickets are difficult to obtain. Each ticket outlet will have its own lottery so that everyone who is in line at a particular outlet to buy tickets when they go on sale has an equal chance of purchasing them. Arlene goes to a ticket outlet where she estimates that her chance of being able to buy tickets is 1/2. Her friend goes to another outlet, where Arlene thinks that her chance of being able to buy tickets is 1/3.

1. What is the probability that both Arlene and her friend are able to buy tickets?

2. What is the probability that neither Arlene nor her friend is able to buy tickets?

3. What is the probability that at least one of the two friends is able to buy tickets?

3.0 A random variable X has the following distribution:

x	-1	0	2	3	4
$P(X = x)$	.1	.3	.2	.1	.3

Find: $P(X > 1)$ $P(X^2 < 2)$

3.0 A fund-raising group sells 1,000 raffle tickets at $5 each. There are three prizes. The first prize is a $1,800 computer. The second prize is a $500 camera, and the third prize is $300 in cash. What is the expected value of a raffle ticket? (ICAS 1997)

3.0 Carla has made an investment of $100. She understands that there is a 50% chance that after a year, her investment will have grown to exactly $150. There is a 20% chance that she will double her money in that year, but there is also a 30% chance that she will lose the entire investment. What is the expected value of her investment after a year?

4.0 You are playing a game in which the probability that you will win is 1/3, and the probability that you will lose or play to a tie is 2/3. If you play this game 8 times, what is the probability that you will win exactly 3 times?

5.0 Suppose that X is a normally distributed random variable with mean μ. Find $P(X < \mu)$.

Advanced Placement Probability and Statistics

1.0 I roll two standard fair dice and look at the numbers showing on the top sides of the two dice. Let *A* be the event that the sum of the two numbers showing is greater than 5. Let *B* be the event that neither die is showing a 1 or a 6. Are events *A* and *B* independent?

5.0 Suppose that *X* is a discrete random variable and that *X* has the following distribution:

x	−1	0	2	5
$P(X = x)$	1/4	1/8	1/2	1/8

Compute the mean for *X.*

6.0 Suppose that *X* is a discrete random variable and that *X* has the following distribution:

x	−1	0	2	5
$P(X = x)$	1/4	1/8	1/2	1/8

Compute the variance for *X.*

9.0 Suppose that a new medical treatment is reported to be successful for 80% of patients. What is the probability that in a sample of 100 patients, 75 or more will find the treatment successful?

12.0 A teacher claims that quiz scores for students are indicative of their test scores. You sample 6 students from this teacher's class and find the following quiz and test scores:

Quiz	_	2	9	6	9	5

scores						
Test scores	85	60	80	70	85	80

Draw a scatter plot for these data, with the quiz scores on the horizontal axis and test scores on the vertical axis. Find the line that best fits these data by using least squares and graph the line along with your scatter plot.

13.0 In the preceding example concerning quiz scores and test scores, by using the graph alone, what can you say about the correlation coefficient? Suppose that 4 more data points are collected and that the best fit line remains approximately the same for the combined data, but that the correlation coefficient now is closer to 1 than it was for just the 6 data points. What can you say about the placement of the 4 additional data points?

16.0 Suppose that it is known that the average lifetime of a particular brand of light bulb is 1,000 hours, with a standard deviation of 90 hours. You sampled 20 of these bulbs and computed that their lifetimes averaged 900 hours, with a sample standard deviation of 120 hours. If you sample another 20 bulbs and combine your data, what is most likely to occur to the average lifetimes for the 40 bulbs and to the sample standard deviation for the 40 bulbs?

17.0 Suppose that the number of cars passing a certain bridge on a freeway during one-minute intervals is normally distributed. Suppose that 61 one-minute observations are randomly made. The average number of cars passing the bridge in a minute, over the 61 observations, is 31. The sample variance is 25. Find a 95% confidence interval for the average number of

cars passing the bridge per minute. For a margin of error of 1, with 95% confidence, how large a sample size would be needed?

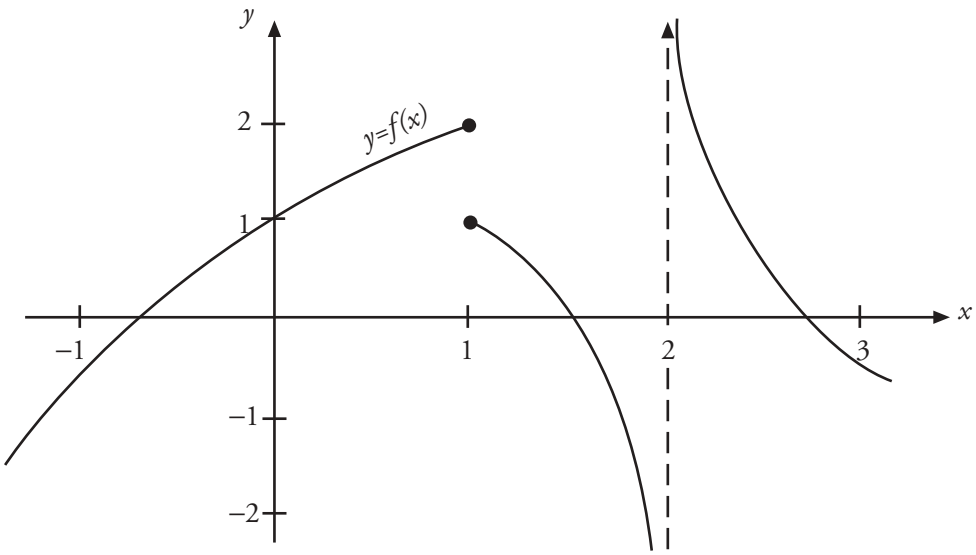
18.0 In the situation described previously, if you were testing the hypothesis that the average number of cars per minute traveling over the bridge is more than 30, what *P*-value would you attach to the data that were collected?

Calculus

1.0 Without using a graphing calculator, evaluate $\lim_{x\to\infty} \arctan x$. Explain what this limit should mean about the graph of the arctangent function and then verify this limit on a graphing calculator.

1.0 Using the graph of *f* shown below, estimate:

$\lim_{x\to 0} f(x)$ $\lim_{x\to 1} f(x)$ $\lim_{x\to 1^-} f(x)$ $\lim_{x\to 2^-} f(x)$



1.0 Using the formal definition of limit, show that $\lim_{x\to 2} (3x+1) = 7$.

1.0 Using the formal definition of limit, show that $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

2.0 Using the formal definitions of continuity and limit, show that $f(x) = 5x + 4$ is continuous.

3.0 Use the Intermediate Value theorem to assert that the equation $4^x = x + 5$ has a solution.

3.0 Give an example that demonstrates that the conclusions of the Intermediate Value theorem need not hold for a function that is not continuous.

8.0 Must $f(x) = \dfrac{|x|}{x+3}$ have a maximum and a minimum value on the interval [–1, 3]? Explain.

4.0 Using the definition of derivative, find the derivative of $f(x) = \sqrt{x+1}$.

4.0 Differentiate:

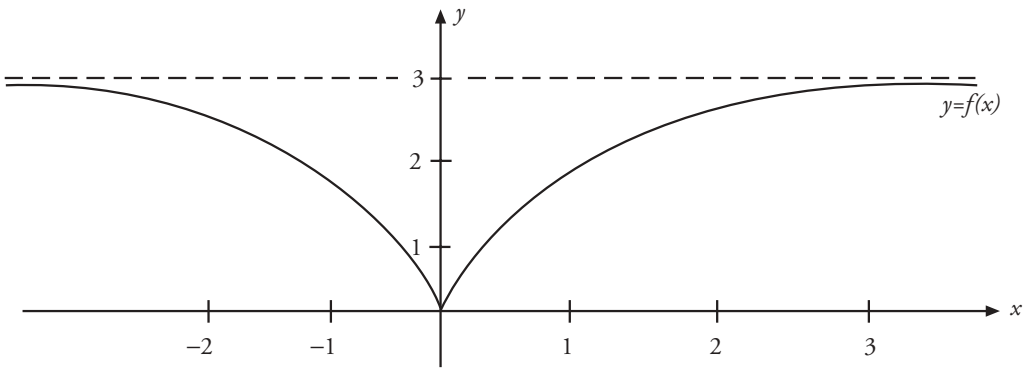
$f(x) = \sin x$

$g(x) = \dfrac{1+\ln x}{e^x}$

$h(x) = \arctan(x)$

4.0 Using the graph of *f* shown below, estimate:

$f'(1) \rightarrow f'(0) \rightarrow \lim_{x \to \infty} f'(x)$



4.0 Find the value of $\lim_{h \to 0} \dfrac{\sqrt{2+h} - \sqrt{2}}{h}$. (Adapted from TIMSS)

4.0 A brush fire spreads so that after t hours, $80t - 20t^2$ acres are burning. What is the rate of growth of the acreage that is burning after 90 minutes?

4.0 According to Newton's law of gravitation, a particle of mass m attracts a particle of mass M with a force whose magnitude is $F = \frac{GmM}{r^2}$, where G is the gravitational constant and r is the distance between the two particles. For particles that are in motion, find the rate of change of F with respect to r.

5.0 Differentiate:

$$g(x) = \ln (x + e^{\cos (x)})$$

$$k(x) = e^{\sqrt{\ln(5-x)}}$$

5.0 Use the facts that $f(x) = \log_2 x$ and $g(x) = 2^x$ are inverse functions and that $g'(x) = 2^x \ln 2$ to find the derivative of $f(x)$.

6.0 A wheel of radius 1 rolls on a straight line (the x-axis) without slipping. The curve traced by a point on the wheel (that starts out on the x-axis) is called a cycloid. The curve can be described parametrically by $x = \theta - \sin \theta$ and $y = 1 - \cos \theta$, where θ is the angle through which the wheel has turned. When the wheel has turned through $\pi/4$ radians, what is $\frac{dy}{dx}$?

7.0 For $f(x) = \arctan x$, find $f'''(x)$.

8.0 Evaluate the following limits:

$$\lim_{x \to 0} \frac{x - \arctan x}{x^3} \qquad \lim_{x \to \infty} \frac{\ln (x)}{\ln (x^2 + 1)} \qquad \lim_{x \to 0^+} (1 + x)^{\csc x}$$

8.0 Use the Mean Value theorem on the following functions, on the given intervals, if it applies:

$f(x) = x + \sin x$ on $[\pi/2, \pi]$

$g(x) = x - x^{2/3}$ on [–1, 1]

8.0 Suppose that f is a continuous function on [a, b], differentiable on (a, b), and that $f'(x) = 0$ for all x in the interval (a, b). Show that f is a constant function on [a, b].

9.0 Without using a graphing calculator, sketch graphs of these functions, showing all local extrema and inflection points:

$g(x) = 3x^4 + 4x^3 + 1$

$h(x) = \ln(1 + x^2)$

10.0 Use Newton's method to approximate a zero for the polynomial $f(x) = x^3 + 3x - 1$ in the interval [0, 1]. You may stop when you have a value of x for which $|f(x)| < 1/1000$.

11.0 A cone is to be made large enough to enclose a cylinder of height 5 and radius 2. What is the smallest possible volume for such a cone?

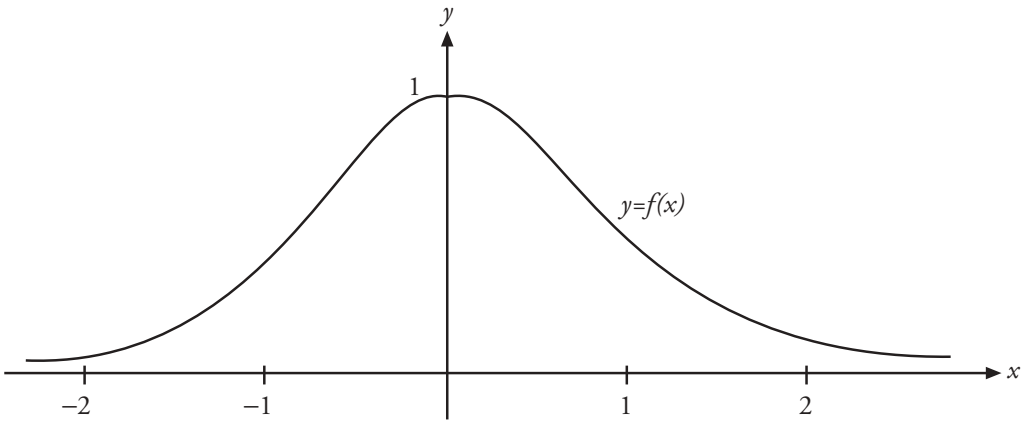
12.0 A climber on one end of a 150-foot rope has fallen down a crevasse and is slipping farther down. This accident happened because his climbing partner, on the other end of the rope, does not have a firm stance. His partner is on the horizontal glacier, slipping toward the crevasse at a rate of 10 ft./sec. At what rate is the distance between the two climbers changing when the first climber is 100 feet down the crevasse?

12.0 A streetlight, 20 feet in height, stands 5 feet from a sidewalk. A person, 6 feet tall, walks along the sidewalk at 4 ft./sec. At what rate is the length of the person's shadow changing when the person is 13 feet from the base of the streetlight?

14.0 A particle moves along a line with velocity function $v(t) = t^3 + t$. Find the distance traveled by the particle between times $t = 0$ and $t = 4$.

14.0 An object thrown upward in a vacuum with initial velocity V_0 will experience an acceleration of -9.8 m/s^2. Use this information to find an expression for the position of the object above its starting position after t seconds.

15.0 On the graph of $f(x) = e^{-x^2}$ shown below, let $g(s)$ denote the area under the graph of f above the x-axis, between $x = 0$ and $x = s$. Find $\lim_{s \to 0} \frac{g(s)}{s}$.



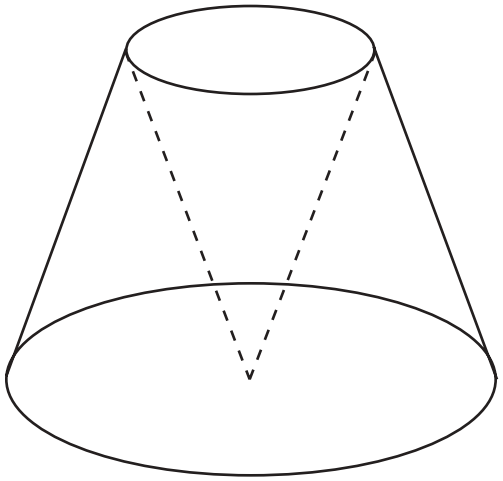
16.0 Consider the region bounded by the x-axis for x> = 1, the line $x = 1$, and the graph of $y = 1/x$. If this region is rotated about the x-axis, find the surface area of the resulting solid.

16.0 Find the length of the curve $y = (4 - x^{2/3})^{3/2}$ between $x = 1$ and $x = 8$.

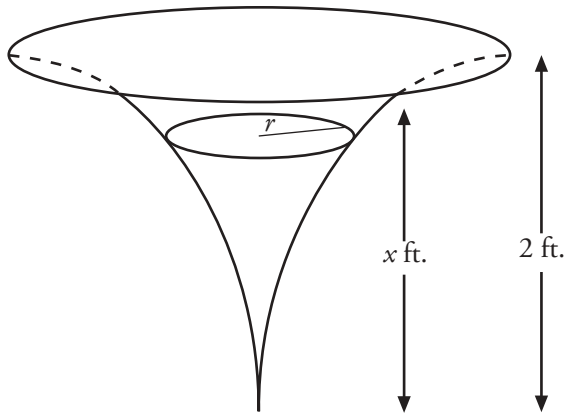
16.0 The following integral represents the volume of a solid that is obtained by rotating a region in the x-y plane about one of the coordinate axes: $\pi \int_0^2 x^4 \, dx$. If this solid was obtained by rotating a region about the x-axis, then what was the region? If, on the other hand, this solid was obtained by rotating a region about the y-axis, what was the region?

16.0 The figure shown below is a cone that has been cut off at the top and then had a cone turned out of it. The radius of the top of the figure is 2 inches.

The radius of the base is 4 inches. The figure is 3 inches tall. Use integration to find the volume of the figure.



16.0 A tank of water is funnel-shaped. The shape of the funnel is such that *x* feet from the base of the tank, the radius of the tank is $r = \sqrt{x}e^{x}$ feet. If the tank is 2 feet deep and full of water, how much work is done in pumping the water out of the tank?



18.0 Evaluate:

sin (arctan (*x*))

tan (arcsin $\sqrt{1-x^2}$)

18.0 Antidifferentiate:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\int \frac{x}{1+x^4}\,dx$$

19.0 Evaluate:

$$\int_1^3 \frac{x^3+6x^2+13x+8}{x^2+4x}\,dx$$

$$\int \frac{x^3+3x-2}{x^2+2x+4}\,dx$$

20.0 Evaluate:

$$\int \frac{\sin x}{\cos^3 x}\,dx$$

$\sin^2 x \cos^4 x dx$

21.0 Estimate $\int_0^3 e^{2x}\,dx$ using Simpson's rule with $n = 6$ subintervals and find a bound on the error.

22.0 Compute the following integrals:

$$\int_0^{\pi} \tan x\,dx$$

$$\int_0^{\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$$

24.0 Find the intervals of convergence for the following power series:

$$\sum_{n=2}^{\infty} \frac{x^n}{3^n\sqrt{n^2+1}} \qquad \sum_{n=1}^{\infty} \frac{(x-5)^n}{n^2-3} \qquad \sum_{n=1}^{\infty} \frac{2^n x^n}{(2n)!}$$

25.0 Use a Maclaurin series for the function $f(x) = \frac{\ln(1+x)}{x}$ to estimate $\int_0^{1/2} \frac{\ln(1+x)}{x}\,dx$ to within .01.

DRAFT: January 28, 2005

10897 **26.0** Find the degree four Taylor polynomial for $f(x) = \sqrt{x}$, centered at $x = 1$.

10898 **26.0** Using the half-angle identity $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$, find a Maclaurin's series

10899 for $f(x) = \sin^2 x$